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Optimal Impulsive Maneuvers to Accomplish Small Plane Changes in an Elliptical Orbit

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This analysis determines the optimal manner in which to perform small plane changes using either a single-impulse or two-impulse maneuver in an arbitrary elliptical orbit. The optimal two-impulse transfer is analytically derived and compared to the familiar single-impulse transfer. Simple expressions for maneuver locations and magnitudes are derived. It is shown that neither type of transfer is always superior to the other. Rather, the condition relating eccentricity to true anomaly at intersection is defined, separating the region in which the single-impulse transfer is superior from the region in which the two-impulse transfer is superior.

Nomenclature

a = semimajor axis
e = orbital eccentricity
f = true anomaly
i = orbital inclination
n = mean orbital motion = $\sqrt{\mu/a^3}$
p = semilatus rectum = $a(1-e^2)$
r = orbital radius = $p/(1+e\cos f)$
u = argument of latitude = $\omega + f$
V_θ = circumferential velocity component = $\sqrt{\mu p/r^2}$
Δi = inclination change
ΔV = velocity change
μ = gravitational constant
ω = argument of perigee
Ω = right ascension of ascending node

Subscripts

1 = in initial orbit
2 = in final orbit
T = total

Superscripts

* = as expressed in the transformed coordinate frame

Introduction

THE problem of transferring between two non-coplanar circular orbits has been examined quite extensively. In Ref. 1 the optimal two-impluse transfer between two such orbits is developed. Reference 2 investigates bielliptic transfers with plane changes distributed among the three impulses and introduces a three-impulse Hohmann-type transfer. A chart covering all possible initial and final circular orbits is also presented, showing the regions in which the various two- and three-impulse transfers yield the lowest characteristic velocity requirements.

For transfers between two non-coplanar, congruent elliptical orbits, solutions have not been so straightforward. Eckel[3-5] has shown that in the general case the optimal two-impulse transfer is given by the solution of three simultaneous equations in three unknowns. Two special cases were studied by Eckel, one in which the line of nodes is perpendicular to the bisector of the angle included between the major axes of the initial and final orbits, and the other in which the line of nodes is parallel to the bisector. In the first special case the two-impulse transfer was described by the solution of two simultaneous equations in two unknowns. In the second case a three-impulse transfer was investigated but found to be inferior to the one- and two-impulse transfers for small angle changes.

Although the optimal three-impulse transfer has not been solved analytically, Niemeier[6] studied the problem of a near-optimal three-impulse transfer between congruent ellipses of any orientation. The three-impulse strategy used was to first circularize at apogee, then perform a plane change with an impulse on the line of nodes, and finally decircularize into the appropriate elliptical orbit. In the case where only a plane change separates the initial and final orbits, a simple one-impulse transfer was found to offer a lower characteristic velocity than the three-impulse transfer for small† plane changes.

The present paper examines the transfer between non-coplanar, congruent elliptical orbits where the two orbits differ by a small angular rotation about a common radius vector. In other words, the size and shape of the two orbits are the same, but their angular momentum vectors differ by a small angle. The three-impulse transfers of Refs. 2, 3, and 6 are inferior to the single-impulse transfer for this case. This study derives analytically the optimal two-impulse transfer for the case of interest and compares it to the single-impulse transfer. Such a plane change solution would be useful for determining the minimum fuel cost of correcting a deviation (whether incurred by injection or perturbation) in a satellite orbit plane.

In general, a plane change maneuver involves changes to the orbital elements: i, Ω, and ω. In order to simplify the problem, a coordinate transformation is introduced, which enables the plane change to be represented in the new coordinates as purely an inclination change maneuver. In this new coordinate frame, the optimal single-impulse transfer and the optimal two-impulse transfer are derived analytically. The resulting equations for maneuver locations and magnitudes are surprisingly simple. In addition, a critical value of orbit eccentricity e is defined, which determines whether the single-impulse or two-impulse transfer should be used in order to achieve the minimum total delta velocity. A summary of the solution equations is presented in Sec. B3 of the Analysis.

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†For example, at an eccentricity of 0.2, the one-impulse transfer is better for plane change angles less than 126 deg. For an eccentricity of 0.7, the plane change angle must be greater than 66 deg for the three-impulse transfer to be of value.

Analysis

A. Coordinate Transformation

Any plane change maneuver can be viewed as purely an "inclination change" maneuver in a transformed coordinate system. In the new coordinate frame, the "equatorial plane" is redefined so as to pass through the intersection points of the initial and final orbits (see Fig. 1). As noted earlier, a general plane change maneuver can involve changes to the orbital elements i, Ω, and ω. Consequently, the true orbital elements of the initial and final orbits will be represented as

Initial orbit:

$$a,e,i_1,\Omega_1,\omega_1 \tag{1a}$$

Final orbit:

$$a,e,i_2,\Omega_2,\omega_2 \tag{1b}$$

The transformed coordinate system is shown in Fig. 1b. In terms of the original orbital elements, the transformed parameters (denoted by an asterisk) become

$$\Delta i_T^* = \cos^{-1}(\cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos\Delta\Omega) \tag{2a}$$

$$\omega^* = \omega_1 - \sin^{-1}(\sin\Delta\Omega \ \sin i_2/\sin\Delta i_T^*) \tag{2b}$$

where $\Delta\Omega = \Omega_2 - \Omega_1$.

Physically, Δi_T^* is the dihedral angle between the two orbit planes and ω^* is the angle from the orbit intersection to perigee in either of the two orbits. Since a, e, and f remain unchanged by this transformation, asterisks will not be used on these quantities in further equations. It should be noted that, in the transformed coordinate frame, the plane change does not cause a change in either the right ascension of ascending node or in the argument of perigee (Ω^* and ω^* are the same in the initial and final orbits).

The inclination change of Eq. (2) can be accomplished by a single impulsive velocity addition at an "equatorial crossing" in the new coordinate frame. The maneuver must occur at an "equatorial crossing" so that a nodal rotation is not introduced. Alternatively, the inclination change can be accomplished using two impulsive velocity additions located so as to cause no net change (after both ΔV's) in the other orbital elements.

B. Optimal Inclination Change Maneuver

1. Single-Impulse Solution

The familiar single-impulse characteristic velocity can be expressed in terms of the circumferential velocity component at an equatorial crossing and the total amount of inclination change to be performed:

$$\Delta V_T = 2V_\theta \sin(\Delta i_T^*/2) \tag{3}$$

Substituting for V_θ yields

$$\Delta V_T = 2\sin\left(\frac{\Delta i_T^*}{2}\right)(1+e\cos f)\sqrt{\frac{\mu}{a(1-e^2)}} \tag{4}$$

To minimize ΔV_T, f is chosen to be at the equatorial crossing of the larger radius, where $(1+e\cos f)$ is smaller, so that $f=-\omega^*$ for 90 deg $<\omega^*<$ 270 deg or $f=$ 180 deg $-\omega^*$ for -90 deg $<\omega^*<$ 90 deg. With the assumption that the inclination change is small, Eq. (4) for the single-impulse ΔV_T takes on the final form

$$\frac{\Delta V_T}{\Delta i_T^*} = \sqrt{\frac{\mu}{a}}\frac{(1-e|\cos\omega^*|)}{\sqrt{1-e^2}} \tag{5}$$

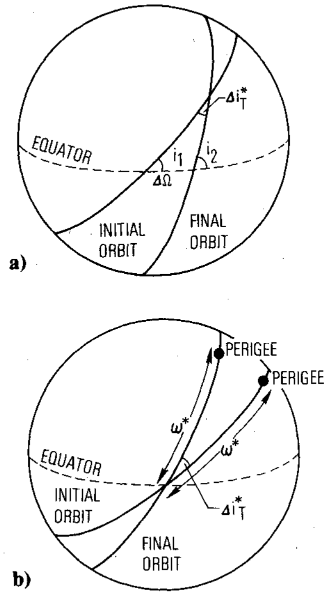

Fig. 1 Geometry of orbital plane change: a) in original coordinate system; b) in transformed coordinate system.

The goal of deriving the single-impluse solution has been achieved. Before turning our attention to the two-impulse solution, let us pursue a worthwhile sidetrack. It will turn out useful to determine the value of e for a fixed a and ω^* which minimizes ΔV_T. This can be done by setting $\partial(\Delta V_T)/\partial e = 0$ in Eq. (4):

$$\frac{\partial \Delta V_T}{\partial e} = 0 = 2\sin\left(\frac{\Delta i_T^*}{2}\right)\sqrt{\frac{\mu}{a}}\frac{e+\cos f}{(1-e^2)^{3/2}} \tag{6}$$

The solution is

$$e = -\cos f$$

where f was chosen at the equatorial crossing having the larger radius. Imposing this condition on f yields

For -90 deg $<\omega^*<$ 90 deg:

$$f = 180° - \omega^* \qquad e = \cos\omega^* \qquad \sqrt{1-e^2} = |\sin\omega^*| \tag{7a}$$

For 90 deg $<\omega^*<$ 270 deg:

$$f = -\omega^* \qquad e = -\cos\omega^* \qquad \sqrt{1-e^2} = |\sin\omega^*| \tag{7b}$$

The solution to Eq. (6) requires that the correction be performed at one of the minor axis points (defined as the points of an orbit where $r=a$). For this value of e, one of the minor axis points is on the "equator." In other words, one of the orbital intersections occurs at a radius equal to the semimajor axis. Let us call the eccentricity defined in Eq. (7) the *critical eccentricity*. At this critical value of e, Eq. (4) can be evaluated (assuming Δi_T^* is small) as

$$\frac{\Delta V_T}{\Delta i_T^*} = (1-e^2)\sqrt{\frac{\mu}{a(1-e^2)}} = \sqrt{\frac{\mu}{a}}\ |\sin\omega^*| \tag{8}$$

2. Two-Impulse Solution

For small changes in inclination, Lagrange's planetary equations can be used:

$$\Delta i_1^* = Kr_1\Delta V_1 \cos u_1^* \tag{9a}$$

$$\Delta\Omega_1^* = Kr_1\Delta V_1(\sin u_1^*/\sin i_1^*) \tag{9b}$$

$$\Delta i_2^* = Kr_2\Delta V_2 \cos u_2^* \tag{9c}$$

$$\Delta\Omega_2^* = Kr_2\Delta V_2(\sin u_2^*/\sin i_2^*) \tag{9d}$$

where

$$K \equiv na/\mu\sqrt{1-e^2}$$

and ΔV_1 and ΔV_2 are normal to their respective orbit planes.

If we assume that $\Delta i_T^* = i_2^* - i_1^*$ is a small quantity, and that i_1^* can be chosen arbitrarily, then we can pick a value for i_1^* such that $\sin i_1^* \approx \sin i_2^*$. Then in order to have no net nodal change, we must require

$$\Delta\Omega_1^* = -\Delta\Omega_2^* \rightarrow r_1\Delta V_1 \sin u_1^* = -r_2\Delta V_2 \sin u_2^* \tag{10}$$

From Eq. (9) we have

$$\Delta V_1 = \Delta i_1^*/Kr_1\cos u_1^* \tag{11}$$

$$\Delta V_2 = \Delta i_2^*/Kr_2\cos u_2^* \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (10) yields the condition

$$\Delta i_1^*/\Delta i_2^* = -\tan u_2^*/\tan u_1^* \tag{13}$$

At this point in the examination of the two-impulse transfer, there are two possibilities: either the two ΔV's have the same direction or they have opposite directions. We will consider both alternatives.

1) ΔV_1 and ΔV_2 in the same direction: In this case the magnitude of the ΔV_T is given by

$$\Delta V_T = \Delta V_1 + \Delta V_2 = \frac{\Delta i_1^*}{Kr_1\cos u_1^*} + \frac{\Delta i_2^*}{Kr_2\cos u_2^*} \tag{14}$$

Substituting $r = p/(1+e\cos f)$, $\Delta i_T^* = \Delta i_1^* + \Delta i_2^*$, and Eq. (13), we find

$$\frac{\Delta V_T}{\Delta i_T^*} = \frac{\sin(\omega^*+f_2)(1+e\cos f_1) - \sin(\omega^*+f_1)(1+e\cos f_2)}{Kp\sin(f_2-f_1)} \tag{15}$$

If we set

$$\partial\left(\frac{\Delta V_T/\Delta i_T^*}{\partial f_1}\right) = 0 \quad \text{and} \quad \partial\left(\frac{\Delta V_T/\Delta i_T^*}{\partial f_2}\right) = 0$$

in order to find a minimum of Eq. (15), we get two simultaneous equations. The solutions to these two equations which also satisfy Eq. (10) are

$$f_1 = f_2 = -\omega^* \qquad u_1^* = u_2^* = 0 \tag{16}$$

and

$$f_1 = f_2 = 180\,\text{deg} - \omega^* \qquad u_1^* = u_2^* = 180\,\text{deg} \tag{17}$$

The solutions presented in Eqs. (16) and (17) are truly minima of $\Delta V_T/\Delta i_T^*$, but they represent the single-impulse solution presented in the previous section, as can be seen by substitution into Eq. (14) and comparison to Eq. (5). Both ΔV_1 and ΔV_2 are applied at the same equatorial crossing and could just as well be combined into a single impulsive maneuver. Results from the earlier investigation of single-impulse maneuvers, including evaluation of $\Delta V_T/\Delta i_T^*$ and critical value of e, fully apply here.

2) ΔV_1 and ΔV_2 in opposing directions: In this case the function of interest is

$$\frac{\Delta V_T}{\Delta i_T^*} = \frac{\Delta V_1 - \Delta V_2}{\Delta i_T^*} = \frac{\sin(\omega^*+f_2)(1-e\cos f_1) + \sin(\omega^*+f_1)(1+e\cos f_2)}{Kp\sin(f_2-f_1)} \tag{18}$$

Again setting the partials of this function with respect to f_1 and f_2 equal to zero yields two simultaneous equations for f_1 and f_2.

The solutions to these two equations which have meaningful results are

$$\cos f_1 = -e \qquad \sin f_1 = \pm\sqrt{1-e^2} \tag{19a}$$

$$\cos f_2 = -e \qquad \sin f_2 = \mp\sqrt{1-e^2} \tag{19b}$$

Notice that these two burns occur at the *two minor axis points* of the ellipse where $r_1 = r_2 = a$.

Substituting Eq. (19) into Eq. (18), we find that the minimum value of $\Delta V_T/\Delta i_T^*$ for this case is given by

$$\frac{\Delta V_T}{\Delta i_T^*} = \frac{\sin(\omega^*+f) + \sin(\omega^*-f)}{Ka\sin(2f)} = \frac{\sin\omega^*}{Ka\sin f}$$

Substituting for K and $\sin f$ yields

$$\Delta V_T/\Delta i_T^* = \sqrt{\mu/a}\,|\sin\omega^*| \tag{20}$$

Notice that this value of $\Delta V_T/\Delta i_T^*$ for the optimal two-impulse solution is independent of e and is equal to the single impulse solution at the critical value of e [Eq. (8)]. Since the single-impulse solution has its lowest value at the critical e, does this mean that the two-impulse solution is always as good as or better than the single-impulse solution [i.e., the best ΔV_T is always given by Eq. (20)]? The answer to this question, as we shall soon see, is no. There are regions for which the two-impulse transfer of Eq. (20) is not available.

Returning to Eq. (10), it is clear that if the sign of ΔV_1 is opposite that of ΔV_2, then $\sin u_1^*$ must be of the same sign as $\sin u_2^*$. Equivalently, the two burns must be accomplished on the same side of the equator. Thus, for the minimum two-impulse solution of Eq. (19) to exist, both the minor axis points must be on the same side of the equator in the transformed coordinate frame. In terms of the original coordinate frame, the two minor axis points must lie on the same side of the line of intersection of the two orbit planes.

Recall that in our investigation of the single-impulse solution, one of the minor axis points occurs at an orbital intersection for the critical eccentricity. For values of eccentricity greater than the critical value, the two minor axis points are on the same side of the line of intersection and the optimal two-impulse solution [Eq. (19)] is available. For values of eccentricity less than the critical value, the minor

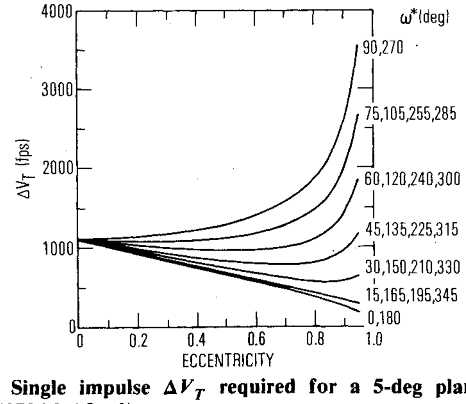

Fig. 2 Single impulse ΔV_T required for a 5-deg plane change, $a = 87149724$ ft ($Q = 2$).

axis points are on opposite sides of the line of intersection. In this region, the optimal two-impulse solutions shown in Eqs. (16) and (17) are available, but these are equivalent to the optimal single-impulse solution of Eq. (7).

3. Solution Summary

The solution for the optimal plane change maneuver can be summarized as follows:

For $e \leq |\cos\omega^*|$, single-impulse transfer at the intersection point of the larger radius is best:

$$\frac{\Delta V_T}{\Delta i_T^*} = \sqrt{\frac{\mu}{a}} \frac{(1 - e|\cos\omega^*|)}{\sqrt{1-e^2}} \tag{21}$$

For $e \geq |\cos\omega^*|$, two-impulse transfer with maneuvers occurring at minor axis points is best:

$$\frac{\Delta V_T}{\Delta i_T^*} = \sqrt{\frac{\mu}{a}} |\sin\omega^*| \tag{22}$$

where the true anomalies of the minor axis points are given in Eq. (19). The velocity split between the two maneuvers is given by

$$\Delta V_1 / \Delta V_2 = -\sin(\omega^* + f_2)/\sin(\omega^* + f_1) \tag{23}$$

and the inclination change (plane change in the original coordinate system) on each maneuver is determined from Eq. (14). The signs of the plane changes and velocity increments in the original coordinate system are chosen so that the plane changes are in the same sense, while the velocity increments are in opposing senses.

The definitions of Δi_T^* and ω^* are given in Eq. (2). Basically, Δi_T^* is the dihedral angle between the two orbit planes and ω^* is the central angle from the line of intersection of the two orbit planes to perigee in either orbit (see Fig. 1b). These equations provide a simple, yet complete, solution for optimal impulsive small plane change maneuvers.

4. Graphical Results

For a variety of ω^* values, the single impulse ΔV_T [Eq. (21)] required to accomplish a 5-deg plane change in a $Q=2$‡ orbit is shown in Fig. 2 as a function of orbit eccentricity. Note that the minimum of each curve is at its respective critical value of e.

For the same values of a, ω^*, and Δi_T^*, Fig. 3 shows the optimal two-impulse ΔV_T [Eq. (22)] as a function of eccentricity. The minimum value of e for which the solution exists is the critical value, and the ΔV_T is constant with respect to e. At the critical value of e, both the single- and two-impulse solutions have the same value. This fact is illustrated more clearly in Fig. 4, where both the single- and double-impulse solutions have been plotted together for two cases. In the case $\omega^* = 90$, 270 deg, the two-impulse solution is always better. In fact, at $e = 0.7$, it is lower by 444 fps (29%). At the other extreme, $\omega^* = 180$ deg, the single-impulse solution would always be lower. The other case, $\omega^* = 45$, 135, 225, 315 deg, shows a transition from the single-impulse solution to the two-impulse solution at $e = 0.707$.

To check the analytical results of Figs. 2-4, a number of cases were run on a computer program that uses a search technique to locate the minimum total ΔV transfer, whether it be a single- or double-impulse maneuver. Results from these computer runs showed excellent agreement with the data in the figures. Additional cases were run for plane changes much larger than 5 deg in order to evaluate the effect of the small angle assumption (inherent in Lagrange's planetary

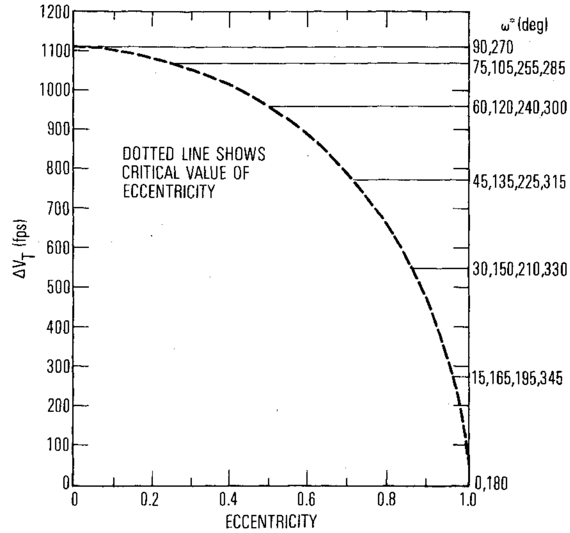

Fig. 3 Optimal two-impulse ΔV_T required for a 5-deg plane change, $a = 87149724$ ft ($Q = 2$).

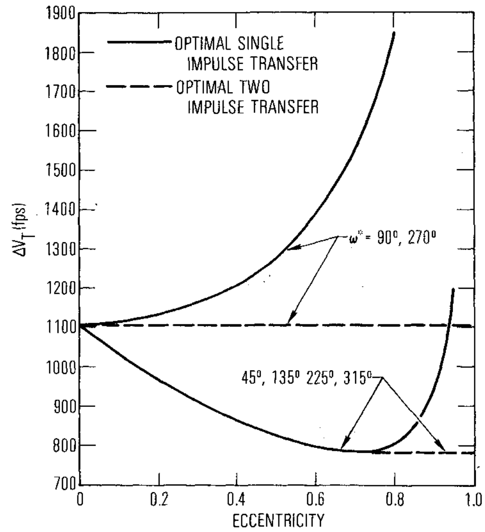

Fig. 4 Single- and two-impulse ΔV_T required for a 5-deg plane change, $a = 87149724$ ft ($Q = 2$).

equations) on the analytical two-impulse solution. For plane changes of 10 deg, the analytical solution differed from the actual optimal two-impulse solution by less than 0.2%. For 20-deg plane changes the difference was less than 0.8%, and for 30-deg plane changes less than 2%. In all cases the analytical solution yielded the higher ΔV_T. It appears that the small angle assumption does not overly restrict the range of accuracy of the analytic solution. Although the data in Figs. 2-4 are for $Q = 2$ orbits only, the analysis is applicable to any circular or elliptical orbit.

Conclusions

A simple analytic solution to the problem of accomplishing a small plane change in an arbitrary elliptical orbit with the minimum total impulsive velocity addition has been obtained. Analysis has shown that the total delta velocity required for either the optimal single-impulse transfer or the optimal two-impulse transfer can be expressed as a simple equation. For the single-impulse transfer, the delta velocity is a function of the orbital eccentricity, semimajor axis and central angle from perigee to the line of intersection of the orbit planes. In the two-impulse transfer, the delta velocity is dependent only on the latter two parameters.

‡$Q = 2$ designates an orbit whose ground trace repeats after two orbital revolutions. The value of a for this orbit is determined by the period, which is approximately twelve hours.

A "critical value" of eccentricity has been defined as that value which places one of the minor axis points of the ellipse on the line of interesection of the orbit plane. For eccentricities less than the critical value, the optimal single-impulse transfer offers the lower total delta velocity. The impulse is applied in a direction normal to the orbit plane at the intersection point of the larger radius. For eccentricities greater than the critical value, the optimal two-impulse transfer is always better. The two burns are performed on the minor axis points of the ellipse in opposing directions normal to the orbit plane.

Acknowledgments

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